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ADDENDUM

Duality for multiparametric quantum $GL(n)$

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Abstract. In our paper (Dobrev V K and Parashar P 1993 *J. Phys. A: Math. Gen.* **26** 6991–7002) we have constructed the Hopf algebra $\mathcal{U}$ as dual to the quantum group deformation of $GL(n)$ (depending on $1 + n(n-1)/2$ parameters), and we have shown that $\mathcal{U}$ has one-generator central Hopf subalgebra and a commutation subalgebra $\mathcal{U}'$ which is a Drinfeld–Jimbo-like deformation of $U(sl(n))$, but which is not a Hopf subalgebra since its coproducts depend also on the central generator. In this addendum we show that $\mathcal{U}'$ is a Hopf subalgebra, besides the trivial case of coinciding parameters, also for a special choice of the parameters, so that there would be only $1 + (n-1)(n-2)/2$ independent parameters.

In the main text [1] we have constructed the Hopf algebra $\mathcal{U} \equiv \mathcal{U}_{uq}$ as the dual to the multiparameter matrix quantum group $GL_{uq}(n)$ (depending on the parameter u and $n(n-1)/2$ parameters $\boldsymbol{q} = \{q_{ij} | j - i > 0\}$) and we have shown that $\mathcal{U}$ may be split in the form: $U_{u,\boldsymbol{q}}(sl(n,\mathbb{C})) \otimes U_u(\mathcal{Z})$, where $\mathcal{Z}$ is one-dimensional spanned by the generator K (cf (29)) and $U_u(\mathcal{Z})$ is a central Hopf subalgebra of $\mathcal{U}$, while $\mathcal{U}' = U_{u,\boldsymbol{q}}(sl(n,\mathbb{C}))$ is a deformation of $U(sl(n,\mathbb{C}))$ which is of Drinfeld–Jimbo form with deformation parameter u as a commutation subalgebra of $\mathcal{U}$. However, $\mathcal{U}'$ is not a Hopf subalgebra of $\mathcal{U}$ since the coproducts of $\mathcal{U}'$ depend also on the generator K. Furthermore, these coproducts depend on all parameters u and $\boldsymbol{q}$. We have also noticed that $\mathcal{U}'$ is a Hopf subalgebra if all parameters coincide.

In this addendum we show that $\mathcal{U}'$ is a Hopf subalgebra also for a special choice of the parameters, so that there would be only $1 + (n-1)(n-2)/2$ independent parameters: u and $\tilde{\boldsymbol{q}} = \{q_{ij} | j-i > 1\}$. In this special case the central generator K decouples from the coproducts of the generators of $\mathcal{U}' = U_{u,\tilde{\boldsymbol{q}}}(sl(n,\mathbb{C}))$ as in the one-parameter deformation, however, these coproducts depend also on $\tilde{\boldsymbol{q}}$.

To make this explicit we first need to express the operators $\mathcal{P}_i$ (cf (43), and through them $\mathcal{Q}_i$, cf (45)) in terms of the generators H_i and K. For this we first express the generators D_i through H_i and K:

$$
\begin{aligned}
D_i &= \frac{1}{n}\left(K - \sum_{j=1}^{i-1} jH_j + \sum_{j=i}^{n-1}(n-j)H_j\right) = \hat{K} + \hat{H}_i \\
\hat{K} &\equiv \frac{1}{n}\left(K - \sum_{j=1}^{n-1} jH_j\right) \qquad \hat{H}_i \equiv \sum_{j=i}^{n-1} H_j \qquad (\hat{H}_n \equiv 0).
\end{aligned}
\tag{A.1}
$$

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Now we substitute (A.1) into (43) to obtain:

$$\mathcal{P}_i = (\tilde{\boldsymbol{q}}_i)^{\hat{K}} \left(\prod_{s=1}^{i-1} \left(\frac{q_{si}}{q_{s,i+1}} \right)^{\hat{H}_s} \right) \left(\frac{u^2}{q_{i,i+1}} \right)^{\hat{H}_i} \left(\frac{1}{q_{i,i+1}} \right)^{\hat{H}_{i+1}} \prod_{t=i+2}^{n-1} \left(\frac{q_{i+1,t}}{q_{it}} \right)^{\hat{H}_t}$$
$$\tilde{\boldsymbol{q}}_i \equiv \left(\prod_{s=1}^{i-1} \frac{q_{si}}{q_{s,i+1}} \right) \frac{u^2}{q^2_{i,i+1}} \prod_{t=i+2}^{n} \frac{q_{i+1,t}}{q_{it}}. \tag{A.2}$$

From the above expression it is clear that in order for K to decouple from the system the $n-1$ constants $\tilde{\boldsymbol{q}}_i$ should become equal to unity. This brings $n-1$ conditions on the parameters q_{ij}. It seems natural to use these conditions to fix the $n-1$ next-to-main-diagonal parameters $q_{i,i+1}$, and indeed, a natural choice for this exists, namely, we may set:

$$q^0_{i,i+1} \equiv u^{i(n-i)} \prod_{\substack{1\leqslant s\leqslant i \quad i+1\leqslant t\leqslant n \\ s<t-1}} q_{st}^{-1}$$
$$= u \widetilde{\prod_{s=1}^{i} \prod_{t=i+1}^{n}} \frac{u}{q_{st}} \qquad 1 \leqslant i \leqslant n-1 \tag{A.3}$$

where the tilde over the double product means that the case $s = i = t - 1$ should be omitted. Then we obtain:

$$(\tilde{\boldsymbol{q}}_i)_{q_{i,i+1}=q^0_{i,i+1}} = 1 \qquad 1 \leqslant i \leqslant n-1 \tag{A.4}$$

and substituting this in the operators $\mathcal{P}_i$ we get in terms of $\hat{H}_i$ and in terms of H_i:

$$\tilde{\mathcal{P}}_i \equiv (\mathcal{P}_i)_{q_{i,i+1}=q^0_{i,i+1}} = \left(\prod_{s=1}^{i-2} \left(\frac{q_{si}}{q_{s,i+1}} \right)^{\hat{H}_s} \right) \left(\frac{u}{q_{i-1,i+1}} \widetilde{\prod_{s=1}^{i-1} \prod_{t=i}^{n}} \frac{u}{q_{st}} \right)^{\hat{H}_{i-1}}$$
$$\times u^{\hat{H}_i - \hat{H}_{i+1}} \left(\widetilde{\prod_{s=1}^{i} \prod_{t=i+1}^{n}} \frac{q_{st}}{u} \right)^{\hat{H}_i + \hat{H}_{i+1}} \left(\frac{u}{q_{i,i+2}} \widetilde{\prod_{s=1}^{i+1} \prod_{t=i+2}^{n}} \frac{u}{q_{st}} \right)^{\hat{H}_{i+2}} \prod_{t=i+3}^{n-1} \left(\frac{q_{i+1,t}}{q_{it}} \right)^{\hat{H}_t}$$
$$= \left(\prod_{j=1}^{i-2} \left(\prod_{s=1}^{j} \frac{q_{si}}{q_{s,i+1}} \right)^{H_j} \right) \left(\left(\prod_{s=1}^{i-1} \frac{u^2}{q^2_{s,i+1}} \right) \prod_{s=1}^{i-1} \prod_{t=i+2}^{n} \frac{u}{q_{st}} \right)^{H_{i-1}}$$
$$\times \left(u \left(\prod_{s=1}^{i-1} \frac{u}{q_{s,i+1}} \right) \prod_{t=i+2}^{n} \frac{q_{it}}{u} \right)^{H_i} \left(\left(\prod_{t=i+2}^{n} \frac{q^2_{it}}{u^2} \right) \prod_{s=1}^{i-1} \prod_{t=i+2}^{n} \frac{q_{st}}{u} \right)^{H_{i+1}}$$
$$\times \left(\prod_{j=i+2}^{n-1} \left(\prod_{t=j+1}^{n} \frac{q_{it}}{q_{i+1,t}} \right)^{H_j} \right). \tag{A.5}$$

Thus, for the particular choice $q_{i,i+1} = q^0_{i,i+1}$ we have the splitting $\mathcal{U}_{u\tilde{q}} \cong U_{u,\tilde{q}}(sl(n, \mathbb{C})) \otimes U_u(\mathcal{Z})$ as tensor product of two Hopf subalgebras. Note that $U_{u,\tilde{q}}(sl(n, \mathbb{C}))$ is a Hopf algebra which is a deformation of $U(sl(n, \mathbb{C}))$, and is of Drinfeld–Jimbo form with deformation parameter u as commutation algebra, but not as a coalgebra since the coproducts depend also on $\tilde{\boldsymbol{q}}$.

References

[1] Dobrev V K and Parashar P 1993 *J. Phys. A: Math. Gen.* **26** 6991–7002